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# The trefoil knot is as universal as it can be

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**Abstract**

Let $\tau_{p,q} \subset S^3$ denote the $p, q$-torus knot. It is known that if $\varphi : M \to S^3$ is a branched covering branched along $\tau_{p,q}$, then $M$ is an orientable Seifert manifold with orientable base and non-zero Euler number. We prove a converse of this fact: If $M$ is an orientable Seifert manifold with orientable base and non-zero Euler number, $M$ is a branched covering of $S^3$ with branching along the knot $\tau_{p,q}$ for any $p, q$, $(p, q) = 1$, and $2 \leqslant p < q$.

As an application of our techniques we compute, in terms of Seifert invariants, the cyclic branched coverings of $S^3$ branched along $\tau_{p,q}$.


*MSC:* 57M12

*Keywords:* Branched covering; Torus knot; Seifert manifold

## 1. Introduction

Since the fundamental paper of Seifert [11] the study of branched coverings of fibered spaces (see Section 2) with branching along fibers has been posed as an interesting question. In particular, the study of branched coverings of the fibrations of the 3-sphere has occupied an ample space in the literature.

Gordon and Heil proved [7] that any covering of the 3-sphere branched along the $p, q$-torus knot is an orientable Seifert manifold with orientable base. Moreover, since the Euler number (see Section 2) of a torus knot Seifert fibering of the 3-sphere is not zero, one has that the Euler number of a covering of the 3-sphere branched along a torus knot is also not zero [9].

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In this work we prove a converse of the previous result, namely, each orientable Seifert manifold with orientable base and non-zero Euler number is a covering of the 3-sphere branched along the $p,q$-torus knot for any $p,q$, $(p,q)=1$, and $2 \leqslant p < q$ (Section 5).

In fact Gordon and Heil proved that a branched covering of *any* Seifert manifold with branching along fibers is also a Seifert manifold. In order to compute the Seifert invariants of branched coverings of an arbitrary fibered space with branching along fibers one can try similar methods and apply the tools developed in the following sections.

One of the favorite themes in the study of branched coverings of fibered spaces is the cyclic coverings of the torus knots. As an application of the developed tools in Section 3, we give, in terms of Seifert invariants, a list of all the cyclic coverings of the $p,q$-torus knot for any $p,q$ (Section 4).

In [11], Seifert obtains homology spheres as cyclic coverings of the 3-sphere branched along the $p,q$-torus knot. He finds that $(n,pq)=1$ is sufficient for the $n$-cyclic covering of the $p,q$-torus knot to be a homology sphere. Subsequently it was verified [1,6,5,8,2,3] that the condition $(n,pq)=1$ is also necessary. We recover this nice result in Section 4 as a Corollary to Theorem 1.

This paper is organized as follows. Section 2 contains some definitions and useful remarks. In Section 3 we develop the main tools for our computations; this section is the technical support for the rest of the paper. In Section 4 we give a list of the cyclic coverings of $S^3$ branched along the $p,q$-torus knot, $\tau_{p,q}$. Finally, Section 5 contains the statement and proof of the Main Theorem (Theorem 2).

## 2. Preliminaries

A *branched covering* between two $n$-manifolds $M$ and $N$ is an open, proper map $\varphi : M \to N$ which is finite-to-one. The usual way to check that an open map $\varphi : M \to N$ is a branched cover is to find a subcomplex $K \subset N$ of codimension two such that the restriction $\varphi| : M - \varphi^{-1}(K) \to N - K$ is a finite covering space; as usual, one asks for the branched covering $\varphi$, that $\varphi|$ does not extend to a covering space map, but it is useful to consider a true covering space as a special case of a branched covering. The subcomplex $K$, usually a submanifold, is called the *branching* of $\varphi$, and $\varphi^{-1}(K)$ is called the *singular set* of $\varphi$. The covering space $\varphi| : M - \varphi^{-1}(K) \to N - K$ is called the *associated covering space* of $\varphi$.

It is known that the associated covering determines the branched covering: Any finite covering space $\psi : M \to N$ can be completed, by a well-known compactification process, to a branched covering $\bar{\psi} : \overline{M} \to \overline{N}$ (see [4]).

Any covering space of $n$ sheets $\psi : X \to Y$, not necessarily a regular covering, defines a representation $\omega_\psi : \pi_1(Y) \to S_n$ into the symmetric group of $n$ symbols, $S_n$, as follows (see [12, p. 8]): Number the preimage of the base point $\psi^{-1}(*) = \{1,2,\ldots,n\}$, and for a class $[\alpha] \in \pi_1(Y)$, consider the liftings $\alpha_1, \alpha_2, \ldots, \alpha_n : I \to X$, where $\alpha_i$ starts at the point $i \in \psi^{-1}(*)$; then define $\omega([\alpha])(i) = \alpha_i(1)$ $(i = 1,2,\ldots,n)$.

Now a homomorphism $\omega : \pi_1(Y) \to S_n$ determines a covering space (connected if and only if $\omega$ is transitive) of $n$ sheets $\psi_\omega : X \to Y$, namely, the covering space corresponding to the subgroup $\omega^{-1}(St(1)) \leqslant \pi_1(Y)$. Any representation $\omega_\psi$ of $\psi$, as in the preceding paragraph, is conjugate to the homomorphism $\omega$ (it only depends on the numbering of $\psi_\omega^{-1}(*)$).

In this paper we will describe a branched covering of a manifold $M$ by giving a codimension two submanifold $K \subset M$ and a representation $\omega : \pi_1(M - K) \to S_n$.

*Seifert manifolds*

Let $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_t, \beta_t$ be integers such that $\alpha_i > 0$ and $(\alpha_i, \beta_i) = 1$ for $i = 1, 2, \ldots, t$. The *Seifert manifold* $M$ associated to the *Seifert symbol* $(O, g; \beta_1/\alpha_1, \beta_2/\alpha_2, \ldots, \beta_t/\alpha_t)$ is constructed as follows:

Let $F$ be an orientable closed surface of genus $g$, and let $D_1, D_2, \ldots, D_t \subset F$ be $t$ disjoint 2-disks. Write $F_0 = \overline{F - \cup D_i}$, and $M_0 = F_0 \times S^1$. If $\partial F_0 = q_1 \sqcup q_2 \sqcup \cdots \sqcup q_t$ and $h = \{x\} \times S^1$ for some $x \in F_0$, we let $m_i \subset q_i \times S^1$ be a simple closed curve such that $m_i \sim q_i^{\alpha_i} h^{\beta_i}$. Let $V_1, V_2, \ldots, V_t$ be solid tori with meridians $\mu_1, \mu_2, \ldots, \mu_t$, respectively, and let $\eta_i : \partial V_i \to q_i \times S^1$ be a homeomorphism such that $\eta_i(\mu_i) = m_i$ for $i = 1, 2, \ldots, t$. Then the Seifert manifold $M$ associated to the symbol $(O, g; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$ is

$$M = M_0 \bigcup_{\bigcup \eta_i} \Bigl(\bigcup V_i\Bigr).$$

The circles $\{x\} \times S^1$ for $x \in F_0$ are called the *ordinary fibers* of $M$ and the core $e_i$ of $V_i$ is called an *exceptional fiber of order* $\alpha_i$ in case $\alpha_i > 1$; otherwise $e_i$ is also an ordinary fiber ($i = 1, 2, \ldots, t$). The surface $F$ is called the *orbit surface* or the *base* of $M$. Note that collapsing each fiber of $M$ into a point gives us an identification $p : M \to F$.

The choices in the construction of $M$ are taken into account by the

**Theorem** [9]. *Two Seifert symbols represent homeomorphic Seifert manifolds by a fiber preserving homeomorphism if and only if one of the symbols can be changed into the other by a finite sequence of the following moves*:

(0) *Permute the ratios.*
(1) *Add or delete* $\frac{0}{1}$.
(2) *Replace the pair* $\frac{\beta_i}{\alpha_i}, \frac{\beta_j}{\alpha_j}$ *by* $\frac{\beta_i + k\alpha_i}{\alpha_i}, \frac{\beta_j - k\alpha_j}{\alpha_j}$.

Notice that the manifold $M$ associated to $(O, g; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$ can be recovered unambiguously from $M_0 = F_0 \times S^1$ and the curves $m_i = q_i^{\alpha_i} h^{\beta_i}$, $i = 1, 2, \ldots, t$. We call the pair $(F_0 \times S^1, \{m_i\})$ a *frame* for $(O, g; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$.

A branched covering of a solid torus $V$ branched along the core $e$ of $V$ is completely determined, up to covering equivalence, by a covering space of $\partial V$. Therefore to describe a branched covering of $(O, g; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$ branched along fibers it suffices to construct a covering space of a frame for $(O, g; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t, 0/1, \ldots, 0/1)$. In this way we obtain a branched covering branched (at most) along the exceptional fibers and the ‘added’ $\frac{0}{1}$-fibers.

From the symbol $(O, g; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$ one can write down a presentation of the fundamental group: Let $a_1, \ldots, a_{2g}$ be a basis of the orbit surface; then

$$\begin{aligned}
&\pi_1(O, g; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t) \\
&\quad\cong \Big\langle a_1, \dots, a_{2g}, q_1, q_2, \dots, q_t, h\colon\ q_1 q_2 \cdots q_t = \prod [a_{2i-1}, a_{2i}], \\
&\qquad\qquad\qquad\qquad [h, a_i] = 1,\ [h, q_j] = 1,\ q_j^{\alpha_j} h^{\beta_j} = 1 \Big\rangle.
\end{aligned}$$

If $(F_0 \times S^1, \{m_i\})$ is a frame for $(O, g; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$ then

$$\begin{aligned}
\pi_1\big(F_0 \times S^1\big) \cong \Big\langle &a_1, \dots, a_{2g}, q_1, q_2, \dots, q_t, h\colon\ [h, a_i] = 1,\ [h, q_j] = 1, \\
&q_1 q_2 \cdots q_t = \prod [a_{2i-1}, a_{2i}] \Big\rangle.
\end{aligned}$$

Then a set of permutations $\omega(a_1), \dots, \omega(a_{2g}), \omega(q_1), \dots, \omega(q_t), \omega(h) \in S_n$ extends to a representation $\omega : \pi_1(F_0 \times S^1) \to S_n$ if and only if $\omega(q_1)\omega(q_2)\cdots\omega(q_t) = \prod [\omega(a_{2i-1}), \omega(a_{2i})]$, $[\omega(h), \omega(a_i)] = (1)$, $[\omega(h), \omega(q_j)] = (1)$ (the symbol (1) is the identity permutation).

All the representations that we will consider in this work will have $\omega(a_i) = (1)$ for $i = 1, 2, \dots, 2g$.

**Lemma 0** [11]. *Let $F$ be a compact orientable surface with $\partial F \neq \emptyset$. If $\varphi : \widetilde{M} \to F \times S^1$ is a finite covering space, then there exists $\widetilde{F} \subset \widetilde{M}$ an orientable surface such that $\widetilde{M} = \widetilde{F} \times S^1$ and $\varphi$ preserves the 'fibers' $\{\tilde{x}\} \times S^1$ for each $\tilde{x} \in \widetilde{F}$.*

**Remark.** In the covering space $\varphi : \widetilde{M} \to F \times S^1$ of Lemma 0 a component $C$ of the preimage $\varphi^{-1}(F)$ can be used as the surface $\widetilde{F}$ if and only if the 'fiber' $\varphi^{-1}(\{a\} \times S^1) = \{x_1\} \times S^1 \sqcup \{x_2\} \times S^1 \sqcup \cdots \sqcup \{x_k\} \times S^1$ intersects $C$ in exactly $k$ points (if and only if $k$ is the number of sheets of $\varphi$). Thus if $\varphi^{-1}(F)$ is connected, then $\widetilde{F} = \varphi^{-1}(F)$ if and only if the fiber $h$ of $F \times S^1$ goes to the identity under the representation $\omega : \pi_1(F \times S^1) \to S_n$ associated to $\varphi$.

If $M = (O, g; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$, then $e(M) = -\sum \frac{\beta_i}{\alpha_i}$ is called de *Euler number of $M$*.

**Theorem** [9]. *Let $M$ and $M'$ be Seifert manifolds with base spaces $F$ and $F'$, and let $f : M \to M'$ be an orientation preserving fiber preserving map. Let the degree of the induced map on a typical fiber be $n$ and the degree of the induced map $\bar{f} : F \to F'$ be $m$. Then $e(M) = (m/n)e(M')$.*

In particular if $\varphi : M \to M'$ is a branched covering and $e(M') \neq 0$, then $e(M) \neq 0$.

In this paper we will denote by $\varepsilon = \varepsilon_n \in S_n$ to the standard $n$-cycle $\varepsilon = (1, 2, \dots, n)$.

## 3. Basic lemmas

Let $(F \times S^1, \{m_i\})$ be a frame for $(O, g; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$; let $\omega : \pi_1(F \times S^1) \to S_n$ be a representation, and let $\varphi : \widetilde{M} \to F \times S^1$ be the covering space associated to $\omega$. If $\{\tilde{m}_j\}$ is a set of components of $\varphi^{-1}(\bigcup m_i)$, one $\tilde{m}_j$ for each component of $\varphi^{-1}(\bigcup (q_i \times S^1))$, then

$(\widetilde{M}, \{\tilde{m}_j\})$ is a frame for some Seifert symbol $(O, \tilde{g}; B_1/A_1, \ldots, B_u/A_u)$; for, by means of a Fox compactification, if we extend $\varphi$ into a branched covering $\bar{\varphi}: \widetilde{M} \bigcup_{\bigcup \tilde{\eta}_j} (\bigcup_{j=1}^{u} \widetilde{V}_j) \to (F \times S^1) \bigcup_{\bigcup \eta_i} (\bigcup_{i=1}^{t} V_i)$, then, for each $j$, $\bar{\varphi}: \widetilde{V}_j \to V_{i_j}$ is a continuous map and, therefore, sends a meridian of $\widetilde{V}_j$ into a meridian of $V_{i_j}$ (more precisely, $\bar{\varphi}^{-1}(m_{i_j}) = \varphi^{-1}(m_{i_j})$ is a set of meridians of $\widetilde{V}_j$); and, by Lemma 0, we know that $\widetilde{M}$ is a product $\widetilde{M} = \widetilde{F} \times S^1$.

Thus, given a representation $\omega: \pi_1(F \times S^1) \to S_n$, the main task for us will be to find out which are the numbers $\tilde{g}, u, A_1, B_1, \ldots, A_u, B_u$ by 'understanding' the preimages of the meridians $\varphi^{-1}(m_i)$. In this section we will compute these numbers for some special representations $\omega: \pi_1(F \times S^1) \to S_n$, namely, representations $\omega$ such that $\omega(h)$ is the identity or $\omega(h)$ is an $n$-cycle.

**Torus Lemma.** *Let $T$ be a torus and let $a, b \subset T$ be a basis for $\pi_1(T)$. Let $\omega: \pi_1(T) \to S_n$ be the representation such that $\omega(a) = \varepsilon^s$ and $\omega(b) = \varepsilon^r$. If $\varphi: \widetilde{T} \to T$ is the covering space associated to $\omega$ and $(n, s) = 1$, then for any integer $s^*$ such that $ss^* \equiv 1 \pmod{n}$, there exists $\tilde{b} \subset \widetilde{T}$ a simple closed curve such that the curves $\tilde{a} = \varphi^{-1}(a)$ and $\tilde{b}$ form a basis for $\pi_1(\widetilde{T})$, and $\varphi(\tilde{a}) = a^n$, and $\varphi(\tilde{b}) = ba^{-s^*r}$.*

**Proof.** Cut $T$ along $a$ and $b$ to obtain a fundamental square $V$ with boundary $b^+, a^+, b^-, a^-$ so that we recover $T$ from $V$ by identifying $b^+$ with $b^-$ and $a^+$ with $a^-$. See Fig. 1.

Let $V(1), V(2), \ldots, V(n)$ be $n$ copies of $V$ such that the boundary of $V(i)$ is $b^+(i), a^+(i), b^-(i), a^-(i)$. Then $\widetilde{T}$ is obtained from $\{V(i)\}$ by identifying $b^+(i)$ with $b^-(\varepsilon^s(i))$ and $a^+(j)$ with $a^-(\varepsilon^r(j))$, $i = 1, 2, \ldots, n$. See Fig. 2.

We let $\tilde{a} \subset \widetilde{T}$ be the preimage of $a$, $\varphi^{-1}(a) = a^+(1) \cup a^+(\varepsilon^s(1)) \cup a^+(\varepsilon^{2s}(1)) \cup \cdots \cup a^+(\varepsilon^{(n-1)s}(1))$. Clearly $\varphi(\tilde{a}) = a^n$ and, therefore, $\tilde{a}$ is an essential simple closed curve on $\widetilde{T}$. If $x \in a^+(1)$, then $x$ is glued with some $y \in a^-(\varepsilon^{ks}(1))$ for some $k$. We let $\tilde{b}$ be a simple 'shortest' path from $y$ to $x$ as in Fig. 2. Then $\tilde{b}$ is a simple closed curve on $\widetilde{T}$ which

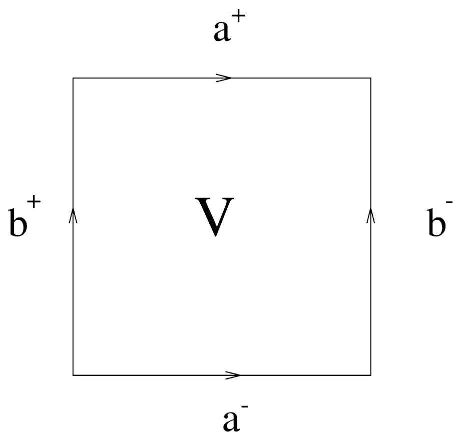


Fig. 1.

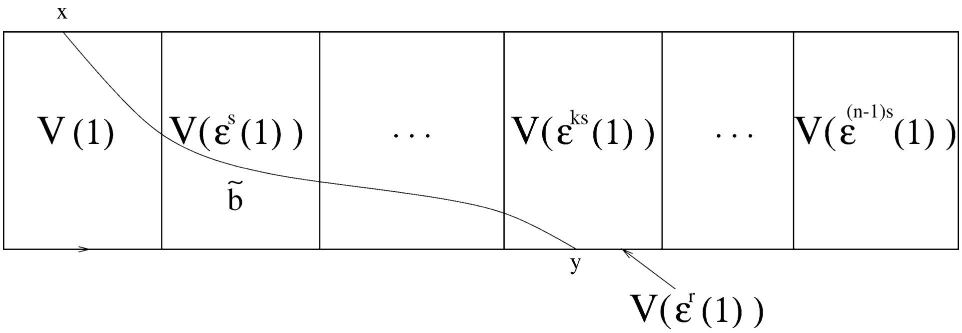


Fig. 2.

intersects $\tilde{a}$ in the single point $x = y \in \widetilde{T}$; therefore $\tilde{a}$ and $\tilde{b}$ is a basis for $\pi_1(\widetilde{T})$. Now, by construction of $\widetilde{T}$, $y \in a^-(\varepsilon^r(1))$; we have that $\varepsilon^{ks} = \varepsilon^r$ and, therefore, $ks \equiv r \pmod n$, or $k \equiv s^*r \pmod n$, where $ss^* \equiv 1 \pmod n$. Since $\tilde{b}$ intersects $k$ vertical lines in Fig. 2 and $\tilde{a}$ in one point, we have that $\varphi(\tilde{b}) = ba^{-k}$ (the sign is a minus because we oriented $\tilde{b}$ from $y$ to $x$); by replacing, if necessary, $\tilde{b}$ with $\tilde{b}\tilde{a}^t$ for some $t$, we may assume that $\tilde{b}$ intersects exactly $s^*r$ vertical lines in Fig. 2 obtaining the required $\varphi(\tilde{b}) = ba^{-s^*r}$. $\square$

**Lemma 1.** *Let $(F \times S^1, \{m_i\})$ be a frame for $(O, g; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$. Let $\omega : \pi_1(F \times S^1) \to S_n$ be the representation such that $\omega(a_j) = (1)$ for $j = 1, \ldots, 2g$; $\omega(q_i) = \varepsilon^{r_i}$ for $i = 1, \ldots, t$ and $\sum r_i = 0$; and $\omega(h) = \varepsilon^s$ with $(n, s) = 1$. And let $s^*$ be any integer such that $ss^* \equiv 1 \pmod n$. If $\varphi : \widetilde{M} \to F \times S^1$ is the covering associated to $\omega$ and $\tilde{m}_i$ is a component of $\varphi^{-1}(m_i)$ $(i = 1, \ldots, t)$, then $(\widetilde{M}, \{\tilde{m}_i\})$ is a frame for $(O, g; B_1/A_1, \ldots, B_t/A_t)$, where*

$$A_i = \frac{n\alpha_i}{d_i}, \qquad B_i = \frac{\beta_i + s^* r_i \alpha_i}{d_i}$$

*and $d_i = (n, \beta_i + s^* r_i \alpha_i)$, $i = 1, \ldots, t$.*

We remark that, by the Torus Lemma, the arbitrary number $s$ in the statement of Lemma 1 can be used for each $i = 1, \ldots, t$ to get the required conclusion as stated.

**Proof.** Let $\widetilde{F} \subset \widetilde{M}$ be the surface such that $\widetilde{M} \cong \widetilde{F} \times S^1$ as in Lemma 0. Since $\varphi| : \varphi^{-1}(F) \to F$ is a (possibly disconnected) covering space of $n$ sheets, we have $\chi(\varphi^{-1}(F)) = n\chi(F)$. Write $\tilde{h} = \varphi^{-1}(h)$; since $\tilde{h}$ intersects $\varphi^{-1}(F)$ in $n$ points (for $\omega(h) = \varepsilon^s$ and $(n, s) = 1$), then the projection $p : \widetilde{M} \to \widetilde{F}$ restricted to $\varphi^{-1}(F)$, $p| : \varphi^{-1}(F) \to \widetilde{F}$, is a map of degree $n$; since there are no exceptional fibers in $\widetilde{M}$, we have that $p| : \varphi^{-1}(F) \to \widetilde{F}$ is a (possibly disconnected) covering space; therefore $\chi(\varphi^{-1}(F)) = n\chi(\widetilde{F})$, and $\chi(F) = \chi(\widetilde{F})$. The number of boundaries of $\widetilde{F}$ equals the number of boundaries of $\widetilde{M}$; since the representation $\omega$ restricted to each $q_i \times S^1$ is transitive ($\omega(h) = \varepsilon^s$ and $(n, s) = 1$), then $\widetilde{M}$ has $t$ boundaries; therefore $F$ and $\widetilde{F}$ have the same number of boundaries and, since both are orientable, we conclude $F \cong \widetilde{F}$.

By the Torus Lemma, there is a curve $\tilde{q}_i \subset \varphi^{-1}(q_i \times S^1)$ such that $\tilde{q}_i$ and $\tilde{h}$ form a basis for $\pi_1(\varphi^{-1}(q_i \times S^1))$, $\varphi(\tilde{h}) \simeq h^n$, and $\varphi(\tilde{q}_i) \simeq q_i h^{-s^* r_i}$ $(ss^* \equiv 1 \pmod n)$; there is a basis $\tilde{a}_1, \ldots, \tilde{a}_{2g} \subset \widetilde{F}$ such that $\varphi(\tilde{a}_i) \simeq a_i$ (for the induced map $\bar{\varphi}: \widetilde{F} \to F$ is a homeomorphism). We compute

$$\begin{aligned}
&\varphi\Big(\tilde{q}_1\tilde{q}_2\cdots\tilde{q}_t\Big(\prod[\tilde{a}_{2i-1}, \tilde{a}_{2i}]\Big)^{-1}\Big) \\
&\quad\simeq q_1h^{-s^*r_1}q_2h^{-s^*r_2}\cdots q_th^{-s^*r_t}\Big(\prod[a_{2i-1}, a_{2i}]\Big)^{-1} \\
&\quad\simeq h^{-s^*\Sigma r_i}q_1q_2\cdots q_t\Big(\prod[a_{2i-1}, a_{2i}]\Big)^{-1} \simeq 1
\end{aligned}$$

(recall $\sum r_i = 0$). Since $\varphi$ is a covering space, the morphism induced by $\varphi$ in fundamental groups is 1–1; therefore $\tilde{q}_1\tilde{q}_2\cdots\tilde{q}_t \simeq \prod[\tilde{a}_{2i-1}, \tilde{a}_{2i}]$ in $\widetilde{M}$; it follows that $\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_t$ span an orbit surface for $\widetilde{M}$ that, without loss of generality, we may assume is $\widetilde{F}$.

Now since $\omega(m_i) = \omega(q_i^{\alpha_i}h^{\beta_i}) = \varepsilon^{\alpha_i r_i + s\beta_i}$ has order $n/d_i$ (recall $d_i = (n, \beta_i + s^*\alpha_i r_i)$), $\omega(m_i)$ has exactly $d_i$ orbits; and, then, $\varphi^{-1}(m_i)$ has $d_i$ components. We see that

$$\varphi(\tilde{m}_i) = m_i^{n/d_i} = q_i^{\alpha_i n/d_i}h^{\beta_i n/d_i},$$

where $\tilde{m}_i$ is a component of $\varphi^{-1}(m_i)$. If $\tilde{m}_i = \tilde{q}_i^{A_i}\tilde{h}^{B_i}$, then

$$\varphi(\tilde{m}_i) = \big(q_ih^{-s^*r_i}\big)^{A_i}\big(h^n\big)^{B_i} = q_i^{A_i}h^{nB_i - s^*r_iA_i};$$

comparing exponents we see that

$$A_i = \frac{\alpha_i n}{d_i} \quad \text{and} \quad B_i = \frac{\beta_i + s^*r_i\alpha_i}{d_i}$$

as claimed. □

**Corollary.** *If $\sum r_i = 0$, and $(n, s) = 1$, and $s^*$ is any integer such that $ss^* \equiv 1 \pmod n$, then $(O, g; B_1/A_1, \ldots, B_t/A_t)$ is an $n$-fold branched covering of $(O, g; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$, where*

$$A_i = \frac{n\alpha_i}{d_i}, \qquad B_i = \frac{\beta_i + s^*r_i\alpha_i}{d_i}$$

*and $d_i = (n, \beta_i + s^*r_i\alpha_i)$, $i = 1, \ldots, t$.*

**Lemma 2.** *Let $(F \times S^1, \{m_i\})$ be a frame for $(O, g; \beta_1/1, \ldots, \beta_u/1, \beta_{u+1}/\alpha_{u+1}, \ldots, \beta_{u+t}/\alpha_{u+t})$ with $\alpha_i > 1$ for $i = u+1, \ldots, u+t$. Let $\omega: \pi_1(F \times S^1) \to S_n$ be a transitive representation such that $\omega(a_i) = (1)$ for $i = 1, \ldots, 2g$; $\omega(q_i) = \varepsilon^{s_i}$ with $(n, s_i) = 1$ for $i = 1, \ldots, u$; $\omega(q_i)$ is a product of disjoint $d_i$-cycles and has $k_i$ fixed points for $i = u+1, \ldots, u+t$; and $\omega(h) = (1)$. If $\varphi: \widetilde{M} \to F \times S^1$ is the covering associated to $\omega$ and $\{\tilde{m}_j\}$ is a set of components of $\varphi^{-1}(\bigcup_{i=1}^{u+t} m_i)$, one $\tilde{m}_j$ for each component of $\varphi^{-1}(\bigcup_{i=1}^{u+t}(q_i \times S^1))$, then $(\widetilde{M}, \{\tilde{m}_j\})$ is a frame for*

$$\Bigl(O, \tilde g; (n\beta_1)/1, \ldots, (n\beta_u)/1, \overbrace{\beta_{u+1}/\alpha_{u+1}, \ldots, \beta_{u+1}/\alpha_{u+1}}^{k_{u+1}\text{-times}},$$
$$\overbrace{\beta_{u+1}/\alpha'_{u+1}, \ldots, \beta_{u+1}/\alpha'_{u+1}}^{n_{u+1}\text{-times}}, \ldots,$$
$$\overbrace{\beta_{u+t}/\alpha_{u+t}, \ldots, \beta_{u+t}/\alpha_{u+t}}^{k_{u+t}\text{-times}}, \overbrace{\beta_{u+t}/\alpha'_{u+t}, \ldots, \beta_{u+t}/\alpha'_{u+t}}^{n_{u+t}\text{-times}}\Bigr),$$

*where $n_i$ is the number of non-trivial orbits of $\omega(q_i)$, $\alpha'_i = \alpha_i/d_i$ $(i = u+1, \ldots, u+t)$, and $\tilde g = (2 - n\chi(F) + u(n-1) + \sum n_i(d_i - 1))/2$.*

**Remark.** The number $d_i$ in the statement of Lemma 2 is a divisor of $\alpha_i$ $(i = u+1, \ldots, u+t)$, for, since $\omega(h) = (1)$, the representation $\omega$ factors through the orbifold group $\langle a_1, \ldots, a_{2g}, q_{u+1}, \ldots, q_{u+t}\colon q_j^{\alpha_j} = 1, q_1 \cdots q_t = \prod[a_{2i-1}, a_{2i}]\rangle$.

**Proof of Lemma 2.** The preimage $\varphi^{-1}(h)$ has $n$ components, hence the restriction of the representation $\omega$ determines a covering $\psi : \widetilde F \to F$ such that $\widetilde M = \widetilde F \times S^1$ and $\varphi = \psi \times 1 : \widetilde M = \widetilde F \times S^1 \to F \times S^1$.

Now since $(n, s_i) = 1$, then $\tilde q_i = \varphi^{-1}(q_i)$ is connected, $i = 1, \ldots, u$; since $\omega(m_i) = \omega(q_i h^{\beta_i}) = \varepsilon^{s_i}$, also $\tilde m_i = \varphi^{-1}(m_i)$ is connected; we have $\varphi(\tilde m_i) = m_i^n = q_i^n h^{n\beta_i}$; if $\tilde m_i = \tilde q_i^{A_i} \tilde h^{B_i}$, then $\varphi(\tilde m_i) = q_i^{nA_i} h^{B_i}$; it follows that $A_i = 1$ and $B_i = n\beta_i$ $(i = 1, \ldots, u)$.

If $i \in \{u+1, \ldots, u+t\}$, then $\varphi^{-1}(q_i)$ has $d_i + k_i$ components $\tilde q_{i,1}, \ldots, \tilde q_{i,n_i}$, $\tilde p_{i,1}, \ldots, \tilde p_{i,k_i}$ where $\varphi| = \psi| : \tilde q_{i,j} \to q_i$ has degree $d_i$, and $\varphi| = \psi| : \tilde p_{i,j} \to q_i$ has degree 1. Therefore the component of $\varphi^{-1}(m_i)$ in $\tilde p_{i,j} \times S^1$ is the curve $\tilde p_{i,j}^{\alpha_i} \tilde h^{\beta_i}$, and a component of $\varphi^{-1}(m_i)$ in $\tilde q_{i,j} \times S^1$ is the curve $\tilde q_{i,j}^{\alpha_i/d_i} \tilde h^{\beta_i}$, for $\omega(m_i) = \omega(q_i^{\alpha_i} h^{\beta_i}) = \omega(q_i)^{\alpha_i} = (1)$.

The expression for $\tilde g$ is a rewriting of the Riemann–Hurwitz formula. □

**Corollary.** *With the numbers $d_i$, $n_i$ and $k_i$ as in Lemma* 2,

$$\Bigl(O, \tilde g; (n\beta_1)/1, \ldots, (n\beta_u)/1, \overbrace{\beta_{u+1}/\alpha_{u+1}, \ldots, \beta_{u+1}/\alpha_{u+1}}^{k_{u+1}\text{-times}},$$
$$\overbrace{\beta_{u+1}/\alpha'_{u+1}, \ldots, \beta_{u+1}/\alpha'_{u+1}}^{n_{u+1}\text{-times}}, \ldots,$$
$$\overbrace{\beta_{u+t}/\alpha_{u+t}, \ldots, \beta_{u+t}/\alpha_{u+t}}^{k_{u+t}\text{-times}}, \overbrace{\beta_{u+t}/\alpha'_{u+t}, \ldots, \beta_{u+t}/\alpha'_{u+t}}^{n_{u+t}\text{-times}}\Bigr)$$

*is an $n$-fold branched covering of $(O, g; \beta_1/1, \ldots, \beta_u/1, \beta_{u+1}/\alpha_{u+1}, \ldots, \beta_{u+t}/\alpha_{u+t})$ where $\alpha_i > 1$ for $i = u+1, \ldots, u+t$, $\alpha'_i = \alpha_i/d_i$ for $i = u+1, \ldots, u+t$, and $\tilde g = (2 - n\chi(F) + u(n-1) + \sum n_i(d_i - 1))/2$.*

## 4. Cyclic coverings of torus knots

Let $\tau_{\alpha_1,\alpha_2} \subset S^3$ be the $\alpha_1,\alpha_2$-torus knot ($2 \leqslant \alpha_1 < \alpha_2$). As an application of the results of Section 3 we will give a list of the cyclic branched coverings of $S^3$ branched along $\tau_{\alpha_1,\alpha_2}$.

First the torus knot $\tau_{\alpha_1,\alpha_2}$ is an ordinary fiber of $(O,0;\beta_1/\alpha_1,\beta_2/\alpha_2)(\cong S^3)$, where $\beta_1\alpha_2+\alpha_1\beta_2=-1$.

Let $(F\times S^1,\{m_0,m_1,m_2\})$ be a frame for $(O,0;0/1,\beta_1/\alpha_1,\beta_2/\alpha_2)$. If $\mu\subset S^3-\tau_{\alpha_1,\alpha_2}$ is a meridian of $\tau_{\alpha_1,\alpha_2}$, then

$$q_0=\mu^{-1},\qquad q_1=a_1^{-\beta_1},\qquad q_2=a_2^{-\beta_2},$$
$$\mu=a_1^{-\beta_1}a_2^{-\beta_2},\qquad h=a_1^{\alpha_1}=a_2^{\alpha_2},$$

where $\pi_1(S^3-\tau_{\alpha_1,\alpha_2})\cong\langle a_1,a_2\colon a_1^{\alpha_1}=a_2^{\alpha_2}\rangle$.

The $n$-fold cyclic covering of $S^3$ branched along $\tau_{\alpha_1,\alpha_2}$ is induced by the representation $\omega:\pi_1(S^3-\tau_{\alpha_1,\alpha_2})\to S_n$ such that $\omega(\mu)=\varepsilon$. If $\omega(a_1)=\varepsilon^{s_1}$ and $\omega(a_2)=\varepsilon^{s_2}$, then $\varepsilon=\omega(\mu)=\omega(a_1^{-\beta_1}a_2^{-\beta_2})=\varepsilon^{-s_1\beta_1-s_2\beta_2}$; so we may assume $s_1=\alpha_2$ and $s_2=\alpha_1$. Therefore the $n$-fold cyclic covering of $S^3$ branched along $\tau_{\alpha_1,\alpha_2}$ is induced by the restriction $\omega:\pi_1(F\times S^1)\to S_n$ such that

$$\begin{aligned}
&\omega(q_0)=\varepsilon^{-1}, &&\omega(m_0)=\omega(q_0)=\varepsilon^{-1},\\
&\omega(q_1)=\varepsilon^{-\alpha_2\beta_1}, &&\omega(m_1)=\omega\big(q_1^{\alpha_1}h^{\beta_1}\big)=(1),\\
&\omega(q_2)=\varepsilon^{-\alpha_1\beta_2}, &&\omega(m_2)=\omega\big(q_1^{\alpha_2}h^{\beta_2}\big)=(1),\\
&\omega(h)=\varepsilon^{\alpha_1\alpha_2}.
\end{aligned}$$

*First case*: $(\alpha_1\alpha_2,n)=1$. Then Lemma 1 applies; here $r_0=-1$, $r_1=-\alpha_2\beta_1$, $r_2=-\alpha_1\beta_2$, and $s=\alpha_1\alpha_2$. We have

$$A_i=\frac{n}{d_i}\alpha_i\quad\text{and}\quad B_i=\frac{\beta_i+(\alpha_1\alpha_2)^*r_i\alpha_i}{d_i},$$

where $d_i=(n,\beta_i+(\alpha_1\alpha_2)^*r_i\alpha_i)$ and $(\alpha_1\alpha_2)(\alpha_1\alpha_2)^*\equiv 1\pmod{n}$. If $k$ is an integer such that $(\alpha_1\alpha_2)(\alpha_1\alpha_2)^*=1-kn$, then $d_0=(n,-(\alpha_1\alpha_2)^*)=1$, $d_1=(n,\beta_1+(\alpha_1\alpha_2)^*\alpha_1\alpha_2\beta_1)=(n,\beta_1kn)=n$, and $d_2=(n,\beta_2+(\alpha_1\alpha_2)^*\alpha_1\alpha_2\beta_2)=(n,\beta_2kn)=n$. Therefore

$$\begin{aligned}
&A_0=n, &&B_0=-(\alpha_1\alpha_2)^*,\\
&A_1=\alpha_1, &&B_1=\beta_1k,\\
&A_2=\alpha_2, &&B_2=\beta_2k.
\end{aligned}$$

That is, in this case

$$\widetilde{M}=\left(O,0;\frac{-(\alpha_1\alpha_2)^*}{n},\frac{\beta_1k}{\alpha_1},\frac{\beta_2k}{\alpha_2}\right)$$

is the $n$-fold cyclic covering of $S^3$ branched along $\tau_{\alpha_1,\alpha_2}$.

Note that $|H_1(\widetilde{M})|=|B_0A_1A_2+A_0B_1A_2+A_0A_1B_2|=|-(\alpha_1\alpha_2)^*\alpha_1\alpha_2+n\beta_1k\alpha_2+n\alpha_1\beta_2k|=|-(1-kn)+nk(\beta_1\alpha_2+\alpha_1\beta_2)|=1$. So that $\widetilde{M}$ is a homology sphere.

*Second case*: $(\alpha_1\alpha_2, n) = n$. Notice that the order $o(\varepsilon^{-\alpha_2\beta_1}) = (\alpha_1, n)$ and $o(\varepsilon^{-\alpha_1\beta_2}) = (\alpha_2, n)$; therefore

$$\omega(q_1) = \varepsilon^{-\alpha_2\beta_1} = u_1u_2\cdots u_{n/(\alpha_1,n)}$$

and

$$\omega(q_2) = \varepsilon^{-\alpha_1\beta_2} = v_1v_2\cdots v_{n/(\alpha_2,n)},$$

where $u_i$ is an $(\alpha_1, n)$-cycle and $v_j$ is an $(\alpha_2, n)$-cycle for each $i$ and $j$. Also $\omega(h) = (1)$, so that Lemma 2 applies. We compute $d_1 = (\alpha_1, n)$, $k_1 = 0$, $n_1 = n/(\alpha_1, n)$, $\alpha_1' = \alpha_1/(\alpha_1, n)$; $d_2 = (\alpha_2, n)$, $k_2 = 0$, $n_2 = n/(\alpha_2, n)$, $\alpha_2' = \alpha_2/(\alpha_2, n)$. The $n$-fold cyclic covering of $S^3$ branched along $\tau_{\alpha_1,\alpha_2}$ is

$$\begin{aligned}\widetilde{M} &= \Bigg(O, \tilde{g}; \frac{0}{1}, \overbrace{\frac{\beta_1}{\alpha_1'}, \ldots, \frac{\beta_1}{\alpha_1'}}^{n_1\text{-times}}, \overbrace{\frac{\beta_2}{\alpha_2'}, \ldots, \frac{\beta_2}{\alpha_2'}}^{n_2\text{-times}}\Bigg)\\ &= \Bigg(O, \tilde{g}; \overbrace{\frac{\beta_1}{\alpha_1'}, \ldots, \frac{\beta_1}{\alpha_1'}}^{n_1\text{-times}}, \overbrace{\frac{\beta_2}{\alpha_2'}, \ldots, \frac{\beta_2}{\alpha_2'}}^{n_2\text{-times}}\Bigg),\end{aligned}$$

where

$$\begin{aligned}\tilde{g} &= \frac{1}{2}\big(2 - 2n + (n-1) + n_1\big((\alpha_1, n) - 1\big) + n_2\big((\alpha_2, n) - 1\big)\big)\\ &= \frac{1}{2}\left(2 - 2n + n - 1 + n - \frac{n}{(\alpha_1, n)} - \frac{n}{(\alpha_2, n)}\right)\\ &= \frac{1}{2}\left(n + 1 - \frac{n}{(\alpha_1, n)} - \frac{n}{(\alpha_2, n)}\right) = \frac{1}{2}\Big(n + 1 - (\alpha_1, n) - (\alpha_2, n)\Big).\end{aligned}$$

Notice that $\tilde{g} = 0$ if and only if $n|\alpha_1$ or $n|\alpha_2$. If, say, $n|\alpha_1$, then the covering is

$$\widetilde{M} = \Big(O, 0; \beta_1/\alpha_1', \overbrace{\beta_2/\alpha_2, \ldots, \beta_2/\alpha_2}^{n\text{-times}}\Big),$$

and we compute

$$\begin{aligned}\big|H_1\big(\widetilde{M}\big)\big| &= \Big|\beta_1\alpha_2^n + \overbrace{\alpha_1'\beta_2\alpha_2^{n-1} + \cdots + \alpha_1'\beta_2\alpha_2^{n-1}}^{n\text{-times}}\Big|\\ &= \big|\beta_1\alpha_2^n + n\alpha_1'\beta_2\alpha_2^{n-1}\big| = \big|\beta_1\alpha_2\alpha_2^{n-1} + \alpha_1\beta_2\alpha_2^{n-1}\big| = \alpha_2^{n-1} > 1,\end{aligned}$$

if $n > 1$. That is, in this case $\widetilde{M}$ is never a homology sphere.

*Third case*: $(\alpha_1, \alpha_2) = d$, with $d \notin \{1, n\}$. In this case $\omega(h)$ is not the identity nor an $n$-cycle, so Section 3 does not apply.

However we can proceed in a similar way as before. We need an extension of the Torus Lemma (whose proof is essentially the same plus some easy and elementary number theory tricks, and is omitted).

**Lemma.** *Let $T$ be a torus and let $a, b \subset T$ be a basis for $\pi_1(T)$. Let $\omega : \pi_1(T) \to S_n$ be the representation such that $\omega(a) = \varepsilon^\sigma$ and $\omega(b) = \varepsilon^\rho$. If $\varphi : \widetilde{T} \to T$ is the covering space associated to $\omega$, then $\widetilde{T}$ is connected if and only if $(n, \rho, \sigma) = 1$. Furthermore,*

*Case* (a): $(n, \rho, \sigma) = 1$. *Let $\tilde{a} \subset \varphi^{-1}(a)$ be a component.*

(a.1) *If $(n, \sigma) = 1$, and $\sigma^*$ is any integer such that $\sigma\sigma^* \equiv 1 \pmod n$, then there exists $\tilde{b} \subset \widetilde{T}$ a simple closed curve such that $\tilde{a}, \tilde{b}$ is a basis for $\pi_1(\widetilde{T})$, $\varphi(\tilde{a}) = a^n$, and $\varphi(\tilde{b}) = ba^{-\sigma^*\rho}$.*

(a.2) *If $(n, \sigma) \neq 1, n$, and $\left(\frac{\sigma}{(n,\sigma)}\right)^*$ is any integer such that $\frac{\sigma}{(n,\sigma)}\left(\frac{\sigma}{(n,\sigma)}\right)^* \equiv 1 \pmod{\frac{n}{(n,\sigma)}}$, then there exists $\tilde{b} \subset \widetilde{T}$ a simple closed curve such that $\tilde{a}, \tilde{b}$ is a basis for $\pi_1(\widetilde{T})$, $\varphi(\tilde{a}) = a^{n/(n,\sigma)}$, and $\varphi(\tilde{b}) = b^{(n,\sigma)}a^{-(\sigma/(n,\sigma))^*\rho}$.*

*Case* (b): $(n, \rho, \sigma) = \zeta \neq 1$. *Then $\widetilde{T}$ has $\zeta$ components $\widetilde{T}_1, \ldots, \widetilde{T}_\zeta$. Let $\tilde{a}_i \subset \varphi^{-1}(a) \cap \widetilde{T}_i$ be a component.*

(b.1) *If $(n, \sigma) = \zeta$, and $\left(\frac{\sigma}{\zeta}\right)^*$ is any integer such that $\frac{\sigma}{\zeta}\left(\frac{\sigma}{\zeta}\right)^* \equiv 1 \pmod{\frac{n}{\zeta}}$, then there exists $\tilde{b}_i \subset \widetilde{T}_i$ a simple closed curve such that $\tilde{a}_i, \tilde{b}_i$ is a basis for $\pi_1(\widetilde{T}_i)$, $\varphi(\tilde{a}_i) = a^{n/\zeta}$, and $\varphi(\tilde{b}_i) = ba^{-(\sigma/\zeta)^*(\rho/\zeta)}$.*

(b.2) *If $(n, \sigma) = \delta \neq \zeta$, and $\left(\frac{\sigma}{\delta}\right)^*$ is any integer such that $\frac{\sigma}{\delta}\left(\frac{\sigma}{\delta}\right)^* \equiv 1 \pmod{\frac{n}{\delta}}$, then there exists $\tilde{b}_i \subset \widetilde{T}_i$ a simple closed curve such that $\tilde{a}_i, \tilde{b}_i$ is a basis for $\pi_1(\widetilde{T}_i)$, $\varphi(\tilde{a}_i) = a^{n/\delta}$, and $\varphi(\tilde{b}_i) = b^{\delta/\zeta}a^{-(\sigma/\delta)^*(\rho/\zeta)}$.*

We are assuming $(\alpha_1\alpha_2, n) = d \neq 1, n$. If $\frac{\alpha_1\alpha_2}{d}(\frac{\alpha_1\alpha_2}{d})^* \equiv 1 \pmod{\frac{n}{d}}$, let $k$ be an integer such that $\frac{\alpha_1\alpha_2}{d}(\frac{\alpha_1\alpha_2}{d})^* = 1 - k\frac{n}{d}$. Call $T_i = q_i \times S^1$, $i = 0, 1, 2$.

Notice that if $\tilde{h}$ is a component of $\varphi^{-1}(h)$, then $\varphi(h) = h^{n/d}$.

For $T_0$, since $\omega(q_0) = \varepsilon^{-1}$, $\widetilde{T}_0$ is connected. By case (a.2) of the lemma, we find $\tilde{q}_0 \subset \widetilde{T}_0$ such that $\varphi(\tilde{q}_0) = q_0^d h^{(\alpha_1\alpha_2/d)^*}$. Since $\omega(m_0) = \varepsilon^{-1}$, we have that $\tilde{m}_0 = \varphi^{-1}(m_0)$ is connected and $\varphi(\tilde{m}_0) = m_0^n$. If, say, $\tilde{m}_0 = \tilde{q}_0^{A_0}\tilde{h}^{B_0}$, we compute $\varphi(\tilde{m}_0) = q_0^{A_0 d} h^{A_0(\alpha_1\alpha_2/d)^* + B_0(n/d)}$; then

$$A_0 = \frac{n}{d} \quad \text{and} \quad B_0 = -\left(\frac{\alpha_1\alpha_2}{d}\right)^*.$$

For $T_1$, the notation of the lemma is $\sigma = \alpha_1\alpha_2$ and $\rho = -\alpha_2\beta_1$; then $(n, \rho, \sigma) = (n, -\alpha_2\beta_1, \alpha_1\alpha_2) = (n, (\alpha_2\beta_1, \alpha_1\alpha_2)) = (n, \alpha_2(\beta_1, \alpha_1)) = (n, \alpha_2) = \zeta_1$. Therefore $\widetilde{T}_1$ is connected if and only if $\zeta_1 = (n, \alpha_2) = 1$. Similarly $\widetilde{T}_2$ is connected if and only if $\zeta_2 = (\alpha_1, n) = 1$.

*First sub-case*: $\zeta_1 = (\alpha_2, n) = 1$. Then $\zeta_2 = (\alpha_1, n) = d$. We have $\widetilde{T}_1$ connected and $\widetilde{T}_2$ with $d$ components $\widetilde{T}_{2,1}, \ldots, \widetilde{T}_{2,d}$.

In $\widetilde{T}_1$, by case (a.2) of the lemma, there exists $\tilde{q}_1 \subset \widetilde{T}_1$ such that $\varphi(\tilde{q}_1) = q_1^d h^{(\alpha_1\alpha_2/d)^*\alpha_2\beta_1}$. Since $\omega(m_1) = (1)$, if $\tilde{m}_1$ is a component of $\varphi^{-1}(m_1)$, then $\varphi(\tilde{m}_1) = m_1 = q_1^{\alpha_1}h^{\beta_1}$. If $\tilde{m}_1 = \tilde{q}_1^{A_1}\tilde{h}^{B_1}$, then $\varphi(\tilde{m}_1) = q_1^{A_1 d} h^{A_1(\alpha_1\alpha_2/d)^*\alpha_2\beta_1 + B_1(n/d)}$. Comparing the exponents we get $\alpha_1 = A_1 d$ or $A_1 = \alpha_1/d$, and $\beta_1 = A_1(\alpha_1\alpha_2/d)^*\alpha_2\beta_1 + B_1(n/d) = (\alpha_1\alpha_2/d)^*(\alpha_1\alpha_2/d)\beta_1 + B_1(n/d) = (1-k(n/d))\beta_1 + B_1(n/d)$, or $B_1 = k\beta_1$.

Now in $\widetilde{T}_{2,i}$, by case (b.2) of the lemma, there exists $\tilde{q}_{2,i} \subset \widetilde{T}_{2,i}$ such that $\varphi(\tilde{q}_{2,i}) = q_2 h^{(\alpha_1\alpha_2/d)^*(\alpha_1\beta_2/d)}$. We have $\varphi(\tilde{m}_{2,i}) = m_2 = q_2^{\alpha_2}h^{\beta_2}$ for $\tilde{m}_{2,i}$ a component of $\varphi^{-1}(m_2) \cap \widetilde{T}_{2,i}$; if $\tilde{m}_{2,i} = \tilde{q}_{2,i}^{A_2}\tilde{h}^{B_2}$, then $\varphi(\tilde{m}_{2,i}) = q_2^{A_2}h^{A_2(\alpha_1\alpha_2/d)^*(\alpha_1\beta_2/d)+B_2(n/d)}$, that is, $A_2 = \alpha_2$ and $B_2 = \beta_2 k$.

By the Riemann–Hurwitz formula, $2 - 2\tilde{g} = 2d - (d-1) - (d-1) - d(1-1) = 2$; therefore

$$\widetilde{M} = \left(O, 0; \frac{-(\alpha_1\alpha_2/d)^*}{(n/d)}, \frac{\beta_1 k}{(\alpha_1/d)}, \overbrace{\frac{\beta_2 k}{\alpha_2}, \ldots, \frac{\beta_2 k}{\alpha_2}}^{d\text{-times}}\right).$$

Notice that $|H_1(\widetilde{M})| = |-(\alpha_1\alpha_2/d)^*(\alpha_1/d)\alpha_2^d + (n/d)\beta_1 k\alpha_2^d + (n/d)(\alpha_1/d)\beta_2 k\alpha_2^{d-1}| = |-(1-k(n/d))\alpha_2^{d-1} + k(n/d)\beta_1\alpha_2\alpha_2^{d-1} + k(n/d)\alpha_1\beta_2\alpha_2^{d-1}| = \alpha_2^{d-1} > 1$, if $d > 1$. Therefore $\widetilde{M}$ is never a homology sphere.

*Second sub-case*: $\zeta_1 = (\alpha_2, n) = d$; then $\zeta_2 = (\alpha_1, n) = 1$. This is symmetric to the first sub-case; then

$$\widetilde{M} = \left(O, 0; \frac{-(\alpha_1\alpha_2/d)^*}{(n/d)}, \overbrace{\frac{\beta_1 k}{\alpha_1}, \ldots, \frac{\beta_1 k}{\alpha_1}}^{d\text{-times}}, \frac{\beta_2 k}{(\alpha_2/d)}\right),$$

and $|H_1(\widetilde{M})| = \alpha_1^{d-1} > 1$. Also $\widetilde{M}$ is not a homology sphere.

*Third sub-case*: $\zeta_1 = (\alpha_2, n) = c_1 \neq 1, d$; then $\zeta_2 = (\alpha_1, n) = c_2 \neq 1, d$ and $d = c_1c_2$. We see that $\widetilde{T}_1$ has $c_1$ components, $\widetilde{T}_{1,1}, \ldots, \widetilde{T}_{1,c_1}$ and $\widetilde{T}_2$ has $c_2$ components $\widetilde{T}_{2,1}, \ldots, \widetilde{T}_{2,c_2}$. By (b.2) of the lemma, there exists $\tilde{q}_{1,i} \subset \widetilde{T}_{1,i}$ such that $\varphi(\tilde{q}_{1,i}) = q_1^{d/c_1}h^{(\alpha_1\alpha_2/d)^*(\alpha_2\beta_1/c_1)}$, and there exists $\tilde{q}_{2,j} \subset \widetilde{T}_{2,j}$ such that $\varphi(\tilde{q}_{2,j}) = q_2^{d/c_2}h^{(\alpha_1\alpha_2/d)^*(\alpha_1\beta_2/c_2)}$.

In $\widetilde{T}_{1,i}$, if $\tilde{m}_{1,i}$ is a component of $\varphi^{-1}(m_1) \cap \widetilde{T}_{1,i}$, then $\varphi(\tilde{m}_{1,i}) = m_1 = q^{\alpha_1}h^{\beta_1}$; if $\tilde{m}_{1,i} = \tilde{q}_{1,i}^{A_1}\tilde{h}^{B_1}$, then $\varphi(\tilde{m}_{1,i}) = q_1^{A_1c_2}h^{A_1(\alpha_1\alpha_2/d)^*(\alpha_2\beta_1/c_1)+B_1(n/d)}$; we obtain $A_1 = \alpha_1/c_2 = \alpha_1/(n, \alpha_1)$, and $B_1 = \beta_1 k$. In $\widetilde{T}_{2,j}$, similarly we get $A_2 = \alpha_2/c_1 = \alpha_2/(n, \alpha_2)$, and $B_2 = \beta_2 k$.

By the Riemann–Hurwitz formula $2 - 2\tilde{g} = 2d - (d-1) - c_1(c_2-1) - c_2(c_1-1) = -c_1c_2 + c_1 + c_2 + 1$; then $\tilde{g} = (c_1c_2 - c_1 - c_2 + 1)/2 = (c_1-1)(c_2-1)/2 \geqslant 1$.

Therefore

$$\widetilde{M} = \left(O, \frac{(c_1-1)(c_2-1)}{2}; \frac{-(\alpha_1\alpha_2/d)^*}{(n/d)},\right.$$
$$\left.\overbrace{\frac{\beta_1 k}{(\alpha_1/(\alpha_1, n))}, \ldots, \frac{\beta_1 k}{(\alpha_1/(\alpha_1, n))}}^{(\alpha_2,n)\text{-times}}, \overbrace{\frac{\beta_2 k}{(\alpha_2/(\alpha_2, n))}, \ldots, \frac{\beta_2 k}{(\alpha_2/(\alpha_2, n))}}^{(\alpha_1,n)\text{-times}}\right)$$

and $\widetilde{M}$ is never a homology sphere. We have proved

**Theorem 1.** *Let $\varphi : \widetilde{M} \to S^3$ be the $n$-fold cyclic branched covering along $\tau_{\alpha_1,\alpha_2}$ and assume $\beta_1\alpha_2 + \alpha_1\beta_2 = -1$.*

(1) *If $(\alpha_1\alpha_2, n) = 1$ and $\alpha_1\alpha_2(\alpha_1\alpha_2)^* = 1 - kn$, then*

$$\widetilde{M} = \left(O, 0; \frac{-(\alpha_1\alpha_2)^*}{n}, \frac{\beta_1 k}{\alpha_1}, \frac{\beta_2 k}{\alpha_2}\right)$$

*and $|H_1(\widetilde{M})| = 1$.*

(2) *If $(\alpha_1\alpha_2, n) = n$, then*

$$\widetilde{M} = \left(O, \tilde{g}; \overbrace{\frac{\beta_1}{\alpha_1'}, \ldots, \frac{\beta_1}{\alpha_1'}}^{n_1\text{-times}}, \overbrace{\frac{\beta_2}{\alpha_2'}, \ldots, \frac{\beta_2}{\alpha_2'}}^{n_2\text{-times}}\right)$$

*and*

$$\left|H_1\big(\widetilde{M}\big)\right| = \begin{cases} \alpha_2^{n-1} & \text{if } n|\alpha_1, \\ \alpha_1^{n-1} & \text{if } n|\alpha_2, \\ \infty & \text{otherwise}, \end{cases}$$

*where $\alpha_i' = \alpha_i/(n, \alpha_i)$, and $n_i = n/(n, \alpha_i)$, $i = 1, 2$, and $\tilde{g} = (n + 1 - (\alpha_1, n) - (\alpha_2, n))/2$.*

(3) *If $(\alpha_1\alpha_2, n) = d \neq 1, n$ and $\frac{\alpha_1\alpha_2}{d}(\frac{\alpha_1\alpha_2}{d})^* = 1 - k\frac{n}{d}$, then*

(a) *If $(\alpha_2, n) = 1$, then*

$$\widetilde{M} = \left(O, 0; \frac{-(\alpha_1\alpha_2/d)^*}{(n/d)}, \frac{\beta_1 k}{(\alpha_1/d)}, \overbrace{\frac{\beta_2 k}{\alpha_2}, \ldots, \frac{\beta_2 k}{\alpha_2}}^{d\text{-times}}\right)$$

*and $|H_1(\widetilde{M})| = \alpha_2^{d-1}$.*

(b) *If $(\alpha_2, n) = d$, then*

$$\widetilde{M} = \left(O, 0; \frac{-(\alpha_1\alpha_2/d)^*}{(n/d)}, \overbrace{\frac{\beta_1 k}{\alpha_1}, \ldots, \frac{\beta_1 k}{\alpha_1}}^{d\text{-times}}, \frac{\beta_2 k}{(\alpha_2/d)}\right)$$

*and $|H_1(\widetilde{M})| = \alpha_1^{d-1}$.*

(c) *If $(\alpha_2, n) \neq 1, d$, then*

$$\widetilde{M} = \Bigg(O, \frac{((\alpha_1, n) - 1)((\alpha_2, n) - 1)}{2}; \frac{-(\alpha_1\alpha_2/d)^*}{(n/d)}, \overbrace{\frac{\beta_1 k}{(\alpha_1/(\alpha_1, n))}, \ldots, \frac{\beta_1 k}{(\alpha_1/(\alpha_1, n))}}^{(\alpha_2,n)\text{-times}}, \overbrace{\frac{\beta_2 k}{(\alpha_2/(\alpha_2, n))}, \ldots, \frac{\beta_2 k}{(\alpha_2/(\alpha_2, n))}}^{(\alpha_1,n)\text{-times}}\Bigg)$$

*and $|H_1(\widetilde{M})| = \infty$.*

**Corollary.** *Let $\varphi : \widetilde{M} \to S^3$ be the $n$-fold cyclic branched covering along $\tau_{\alpha_1,\alpha_2}$, then $\widetilde{M}$ is a homology sphere if and only if $(\alpha_1\alpha_2, n) = 1$.*

## 5. Proof of the main theorem

In this section $\tau_{p,q} \subset S^3$ will denote the $p, q$-torus knot with $2 \leqslant p < q$. If $k \subset S^3$ is a knot, the phrase '$\varphi : M \to (S^3, k)$ is a branched covering' will mean '$\varphi : M \to S^3$ is a branched covering branched along $k$'. We write $|X|$ for the number of components of the space $X$.

**Lemma 3** [10]. *There exists $\varphi : (O, 0; \pm 1/1) \to (S^3, \tau_{p,q})$ a $pq$-fold branched covering with $|\varphi^{-1}(\tau_{p,q})| = (p-1)(q-1)+1$.*

**Remark.** In [10] one has $\varphi : (O, 0; 1/1) \to (S^3, \tau_{p,q})$ and $\bar{\varphi} : (O, 0; -1/1) \to (S^3, \bar{\tau}_{p,q})$ fiber preserving $pq$-fold branched coverings, where $\bar{\tau}_{p,q}$ is the mirror image of $\tau_{p,q}$; if $\eta : (S^3, \bar{\tau}_{p,q}) \to (S^3, \tau_{p,q})$ is a reflection, then $\eta \circ \bar{\varphi} : (O, 0; -1/1) \to (S^3, \tau_{p,q})$ gives us a (non-fiber preserving) $pq$-fold branched covering with $|\bar{\varphi}^{-1}(\bar{\tau}_{p,q})| = |(\eta \circ \bar{\varphi})^{-1}(\tau_{p,q})|$.

**Lemma 4.** *If $k > 0$ and $\mu$ is a non-zero integer, then there exists a branched covering $\varphi : (O, 0; \mu/1) \to (S^3, \tau_{p,q})$ with $|\varphi^{-1}(\tau_{p,q})| > k$.*

**Proof.** Assume $\mu > 0$. By Lemma 3 we have $\varphi_1 : (O, 0; 1/1) \to (S^3, \tau_{p,q})$ a branched covering with $|\varphi_1^{-1}(\tau_{p,q})| \geqslant 3$ (the 'worst' case is $|\varphi_1^{-1}(\tau_{2,3})| = 3$).

Let $(F \times S^1, \{m_0, m_1, m_2\})$ be a frame for $(O, 0; 1/1, 0/1, 0/1)$, where the $\frac{0}{1}$-fibers $e_1, e_2 \subset (O, 0; 1/1)$ are preimages of $\tau_{p,q}$, and let $\omega_2 : \pi_1(F \times S^1) \to S_{\mu k}$ be the representation such that $\omega_2(q_0) = (1)$, $\omega_2(q_1) = \varepsilon$, $\omega_2(q_2) = \varepsilon^{-1}$, and $\omega_2(h) = (1)$. If $\varphi_2 : \widetilde{M} \to (O, 0; 1/1)$ is the covering corresponding to $\omega_2$, then, by Lemma 2,

$$\widetilde{M} = \Bigl(O, 0; \overbrace{1/1, \ldots, 1/1}^{\mu k\text{-times}}, 0/1, 0/1\Bigr) = \bigl(O, 0, (\mu k)/1\bigr)$$

(for $\tilde{g} = (2 - 2\mu k + 2(\mu k - 1) + (1-1))/2 = 0$), and $|\varphi_2^{-1}(h)| = \mu k > k$ for any fiber $h \subset F \times S^1$ (in particular $|\varphi_2^{-1}(\varphi_1^{-1}(\tau_{p,q}))| > k$, for $e_3 \subset \varphi_1^{-1}(\tau_{p,q}) - e_1 \cup e_2$ is a fiber of $F \times S^1$). If $\omega_3 : \pi_1(O, 0; (\mu k)/1) \to S_k$ is the representation such that $\omega_3(q_0) = (1)$, and $\omega_3(h) = \varepsilon_k$, then $\varphi_3 : \widetilde{N} \to (O, 0; (\mu k)/1)$, the associated covering of $\omega_3$, is a covering space with $\widetilde{N} = (O, 0; \mu/1)$. Also $|\varphi_3^{-1}(h)| = 1$ for any fiber $h$. Then $\varphi = \varphi_1\varphi_2\varphi_3 : (O, 0; \mu/1) \to (S^3, \tau_{p,q})$ is a branched covering with $|\varphi^{-1}(\tau_{p,q})| > k$. The case $\mu < 0$ is handled analogously. □

**Lemma 5.** *If $\mu$ is a non-zero integer and $g \geqslant 0$, then there exists a branched covering $\psi : (O, g; \mu/1) \to (S^3, \tau_{p,q})$.*

**Proof.** Let $e_1, e_2, \ldots, e_{2(g+1)} \subset \varphi^{-1}(\tau_{p,q})$ be components, where the covering

$$\varphi : (O, 0; \mu/1) \to \bigl(S^3, \tau_{p,q}\bigr)$$

is as in Lemma 4.

Let $(F \times S^1, \{m_0, m_1, \ldots, m_{2(g+1)}\})$ be a frame for $(O, 0; \mu/1, 0/1, \ldots, 0/1)$, where the $\frac{0}{1}$–fibers are the curves $e_1, \ldots, e_{2(g+1)}$. Let $\omega_1 : F \times S^1 \to S_2$ be the representation such that $\omega_1(q_0) = (1)$, $\omega_1(q_i) = (1, 2)$ for $i = 1, \ldots, 2(g+1)$, and $\omega_1(h) = (1)$, and let $\varphi_1 : \widetilde{M} \to (O, 0; \mu/1)$ be the associated covering to $\omega_1$. Then, by Lemma 2, $\widetilde{M} = (O, \tilde{g}; \mu/1, \mu/1)$ with $\tilde{g} = (2 - 2\cdot 2 + 2(g+1)(2-1))/2 = g$; that is, $\widetilde{M} = (O, g; (2\mu)/1)$. By taking the double covering of $\widetilde{M}$ with representation $\omega_3$ such that $\omega_3(q_0) = (1)$ and $\omega_3(h) = (1, 2)$, we obtain, after composing, a branched covering $\psi : (O, g; \mu/1) \to (S^3, \tau_{p,q})$ as desired. □

**Theorem 2.** *If $e = \sum_{i=1}^{t} \frac{\beta_i}{\alpha_i} \neq 0$, then $(O, g; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$ is a branched covering of $(S^3, \tau_{p,q})$.*

**Proof.** By induction on $t$. Assume $t = 1$. If $\alpha_1 = 1$, then Lemma 5 gives us the desired covering. So assume $\alpha_1 > 1$, and let $(F \times S^1, \{m_0, m_1\})$ be a frame for $(O, g; 0/1, \beta_1/1)$. Let $\omega : \pi_1(F \times S^1) \to S_{\alpha_1}$ be the representation such that $\omega(q_0) = \varepsilon$, $\omega(q_1) = \varepsilon^{-1}$, and $\omega(h) = \varepsilon^{\beta_1^*}$, where $\beta_1 \beta_1^* \equiv 1 \pmod{\alpha_1}$. If $\varphi : \widetilde{M} \to (O, g; \beta_1/1)$ is the branched covering associated to $\omega$, then, by Lemma 1, we have that $\widetilde{M} = (O, g; B_0/A_0, B_1/A_1)$ with $A_0 = \alpha_1$, $B_0 = 0 + (\beta_1^*)^*(1)(1) = \beta_1$, and $A_1 = 1$, $B_1 = \beta_1 + (\beta_1^*)^*(-1) = 0$, that is $\widetilde{M} = (O, g; \beta_1/\alpha_1, 0/1) = (O, g; \beta_1/\alpha_1)$. Notice that the branching of $\varphi$ is along the $(\beta_1/1)$-fiber and the $\frac{0}{1}$-fiber which we may assume are preimages of $\tau_{p,q}$ in the covering of Lemma 5.

Assume that the theorem is true for $t$; that is, if $\sum_{i=1}^{t} \frac{\beta_i}{\alpha_i} \neq 0$, then the manifold $(O, g; \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$ is a branched covering of $(S^3, \tau_{p,q})$.

Assume now that $e = \sum \frac{\beta_i}{\alpha_i} + \frac{\beta}{\alpha} \neq 0$ and let $i \in \{1, 2, \ldots, t\}$.

(1) If $(\alpha_i, \alpha) = 1$, choose integers $r_i, b_i$ such that $(\alpha\beta_i)/\alpha_i = r_i + (b_i/\alpha_i)$, and write $a_i = \alpha_i$.
(2) If $(\alpha_i, \alpha) = d_i$ with $d_i \notin \{1, \alpha\}$, write $\alpha_i' = \alpha_i/d_i$, $\alpha' = \alpha/d_i$, and choose integers $r_i, b_i$ such that $(\alpha'\beta_i)/\alpha_i' = r_i + (b_i/\alpha_i')$. Note that $(\alpha'\beta_i, \alpha) = \alpha'$. Write also $a_i = \alpha_i'$.
(3) If $(\alpha_i, \alpha) = \alpha$, write $d_i = \alpha_i/\alpha$, $r_i = 1$, $b_i = \beta_i - d_i$ and $a_i = d_i$.
(4) Write $r_0 = -\sum_{i=1}^{t} r_i$, $b_0 = \beta + \sum_{i=1}^{t} r_i$, and $a_0 = 1$.

Let $(F \times S^1, \{m_0, m_1, \ldots, m_t\})$ be a frame for $(O, g; b_0/a_0, b_1/a_1, \ldots, b_t/a_t)$, and let $\omega : \pi_1(F \times S^1) \to S_\alpha$ be the representation such that $\omega(v_j) = (1)$ for $i = 1, \ldots, 2g$ and $\{v_j\}$ a basis for $\pi_1(F)$; $\omega(q_i) = \varepsilon^{r_i}$, $i = 0, 1, \ldots, t$; and $\omega(h) = \varepsilon$. By Lemma 1, if $\varphi : \widetilde{M} \to (O, g; b_0/a_0, b_1/a_1, \ldots, b_t/a_t)$ is the associated branched covering of $\omega$, then $\widetilde{M} = (O, g; B_0/A_0, B_1/A_1, \ldots, B_t/A_t)$ where

(1) If $(\alpha_i, \alpha) = 1$, then $(\alpha, b_i + r_i a_i) = (\alpha, \alpha\beta_1) = \alpha$ (by construction of $b_i$ and $r_i$), so that $A_i = (\alpha/\alpha)a_i = \alpha_i$ and $B_i = (b_i + r_i a_i)/\alpha = (\alpha\beta_i)/\alpha = \beta_i$.
(2) If $(\alpha_i, \alpha) = d_i$ with $d_i \notin \{1, \alpha\}$, then $(\alpha, b_i + r_i a_i) = (\alpha, \alpha'\beta_i) = \alpha'$, so that $A_i = (\alpha/\alpha')a_i = d_i(\alpha_i/d_i) = \alpha_i$ and $B_i = (b_i + r_i a_i)/\alpha' = (\alpha'\beta)/\alpha' = \beta_i$.
(3) If $(\alpha_i, \alpha) = \alpha$, then $(\alpha, b_i + r_i a_i) = (\alpha, \beta_i - d_i + d_i) = (\alpha, \beta_i) = 1$, for $(\alpha_i, \beta_i) = 1$ and $\alpha | \alpha_i$; so that $A_i = (\alpha/1)a_i = \alpha(\alpha_i/\alpha) = \alpha_i$ and $B_i = b_i + r_i a_i = \beta_i - d_i + d_i = \beta_i$.

(4) $A_0 = (\alpha/(\alpha, a_0))a_0 = \alpha$ and $B_0 = b_0 + (1)^* r_0 a_0 = (\beta + \sum r_i) - \sum r_i = \beta$.

Therefore $\widetilde{M} = (O, g; \beta/\alpha, \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$ is a branched covering of the manifold $(O, g; b_0/a_0, b_1/a_1, \ldots, b_t/a_t)$; notice that the branching which appears in this construction is (at most) in the $(b_i/a_i)$-fibers. One computes easily that (as it should be; see the last theorem of Section 2) $\sum_{i=0}^{t} \frac{b_i}{a_i} = \alpha(\sum_{i=1}^{t} \frac{\beta_i}{\alpha_i} + \frac{\beta}{\alpha}) \neq 0$; therefore, by induction, $(O, g; b_0/a_0, b_1/a_1, \ldots, b_t/a_t)$ is a branched covering of $(S^3, \tau_{p,q})$, and, with this, we conclude that also the manifold $(O, g; \beta/\alpha, \beta_1/\alpha_1, \ldots, \beta_t/\alpha_t)$ is a branched covering of $(S^3, \tau_{p,q})$. □

**Example.** Following the construction of the proof of Theorem 2 we see that $(O, 0; -1/2, 1/3, 1/5)$ is a branched covering of $(O, 0; -1/2, 2/3)$ with a 5-fold covering, for $\frac{5(-1)}{2} = -2 + \frac{-1}{2}$ and $\frac{5(1)}{3} = 1 + \frac{2}{3}$. Now $(O, 0; -1/2, 2/3)$ is a branched covering of $(O, 0; -1/2)$ with a 3-fold covering, for $\frac{3(-1)}{2} = -1 + \frac{-1}{2}$. And, by the first induction step, $(O, 0; -1/2)$ is a branched covering of $(O, 0; -1/1)$ with a 2-fold covering. By, say, Lemma 3, $(O, 0; -1/1)$ is a $pq$-fold covering of $(S^3, \tau_{p,q})$.

That is, the Poincaré homology sphere $(O, 0; -1/2, 1/3, 1/5)$ is a branched covering of $(S^3, \tau_{p,q})$ with a $30pq$-fold covering.

Notice that, for the trefoil knot, in the first step of this example we obtained a 5-fold covering (the 5-fold cyclic covering) of $(O, 0; -1/2, 1/3, 1/5)$ over $(S^3, \tau_{2,3})$; and following the proof of Theorem 2 we get a 180-fold covering of $(O, 0; -1/2, 1/3, 1/5)$ over $(S^3, \tau_{2,3})$.

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